

# ON THE CRITICAL PAIR THEORY IN ABELIAN GROUPS: BEYOND CHOWLA’S THEOREM

YAHYA O. HAMIDOUNE, ORIOL SERRA\*, GILLES ZÉMOR

Received December 15, 2005

We obtain critical pair theorems for subsets  $S$  and  $T$  of an abelian group such that  $|S+T| \leq |S|+|T|$ . We generalize some results of Chowla, Vosper, Kemperman and a more recent result due to Rødseth and one of the authors.

## 1. Introduction

Let  $S$  and  $T$  be nonempty subsets of  $\mathbb{Z}/p\mathbb{Z}$ . The Cauchy–Davenport Theorem [2, 4] states that

$$|S + T| \geq \min(p, |S| + |T| - 1).$$

The Cauchy–Davenport Theorem was generalized to abelian groups by several authors including Mann [20] and Kneser [19]. The first generalization to cyclic groups is due to Chowla [3]; it states:

**Theorem 1.** *Let  $S, T$  be nonempty subsets of  $\mathbb{Z}/n\mathbb{Z}$  such  $0 \in S$ . Assume that every element of  $S \setminus \{0\}$  has order exactly  $n$ . Then  $|T+S| \geq \min(n, |S|+|T|-1)$ .*

Subsets achieving equality in an additive theorem are known as critical pairs of the theorem. One may easily check that the only interesting critical pairs for the Cauchy–Davenport Theorem arise when  $|S|, |T| \geq 2$  and

---

*Mathematics Subject Classification (2000):* 11P70

\* Supported by the Spanish Research Council under project MTM2005-08990-C02-01 and by the Catalan Research Council under project 2005SGR00256.

$|S+T| \leq p-2$ . Under these assumptions Vosper's Theorem [24] states that  $|S+T| = |S| + |T|$  unless both  $S$  and  $T$  are arithmetic progressions with a common difference. This statement determines the critical pairs of the Cauchy–Davenport Theorem.

Generalizing Vosper's Theorem to arbitrary abelian groups requires a lot of care. The importance of this question was mentioned by Kneser in [19]. Motivated by Kneser's work, Kemperman proposed in [18] a recursive procedure which generalizes Vosper's Theorem to abelian groups. The main tools used by Kemperman are basic transformations introduced by Cauchy, Davenport and Dyson [21]. One of the results obtained by Kemperman is the following:

**Theorem 2 (Kemperman, [18]).** *Let  $G$  be a finite abelian group and let  $S, T$  be subsets of  $G$  such that  $|S| \geq 2$ ,  $|T| \geq 2$  and  $|S+T| = |S| + |T| - 1 \leq p-2$ , where  $p$  is the smallest prime divisor of  $|G|$ . Then  $S$  and  $T$  are arithmetic progressions with the same difference.*

Note that the existence of a short direct proof for this result is unlikely since the statement contains Vosper's Theorem. This result has been recently extended to non abelian groups by Károlyi [17] and independently by one of the authors [8, Theorem 3.2].

By using the additive transformations mentioned above, Rødseth and one of the authors recently characterized the critical pairs of Vosper's Theorem [10]:

**Theorem 3.** *Let  $S, T$  be subsets of a group of prime order  $\mathbb{Z}/p\mathbb{Z}$ , with  $|T| \geq 3$  and  $|S| \geq 4$  such that*

$$|S+T| = |S| + |T| \leq p-4.$$

*Then  $S$  and  $T$  are included in arithmetic progressions with the same difference and of respective lengths  $|S|+1$  and  $|T|+1$ .*

There are several methods currently available in additive theory. One of them is based on Fourier analysis. Examples of applications of this method can be found in the monographs of Freiman [15] and Tao and Vu [23], or in the papers by Deshouillers and Freiman [5], and by Green and Ruzsa [6]. Another powerful tool is the polynomial method introduced by Alon, Nathanson and Ruzsa [1]. Károlyi recently [16] used this method to obtain a remarkable critical pair theorem for restricted sums.

In this paper we obtain improvements of some of the above results using the *isoperimetric* method. This method has been used to generalize addition theorems to non abelian groups in some papers including [25, 12, 8, 11]. It also

derives additive inequalities, mainly from the structure of the  $k$ -atoms of a set. If  $S$  is a generating subset containing 0 of an abelian group  $G$ , a set  $A$  is called a  $k$ -atom of  $S$  if it is of minimum cardinality among subsets  $X$  such that  $|X| \geq k$ ,  $|X+S| \leq |G|-k$ , and  $|X+S|-|X|$  is of minimum possible cardinality (see [Section 2](#) for detailed definitions). It is proved in [\[7\]](#) that any 1-atom containing 0 is a subgroup. This result implies easily Mann's generalization of the Cauchy–Davenport Theorem. The structure of 2-atoms has proved more difficult to describe but potentially gives stronger results: 2-atoms have been used in [\[9, 13\]](#) to derive critical pair results. In groups of prime order, the description of 2-atoms was completed by two of the present authors in [\[22\]](#). Atoms of higher order were used in [\[14\]](#) to classify sets  $S, T \subset \mathbb{Z}/p\mathbb{Z}$  with  $|S+T| \leq |S|+|T|+1$ .

In the present paper we first study the structure of 2-atoms in general abelian groups. Our main result in the first part of this paper is [Theorem 21](#): broadly speaking it states that, under some technical conditions that will be shown to be quite tight, 2-atoms have cardinality 2 or are subgroups. In the rest of the paper we apply this fact to obtain critical pair results.

We shall first obtain a critical pair result for Chowla's [Theorem 1](#) which reduces to Vosper's Theorem if  $n$  is a prime. To be precise, we will actually be dealing with a strengthened version of [Theorem 1](#) ([Corollary 8](#)) that only requires the order of every element of  $S \setminus \{0\}$  to exceed  $|S|-1$  rather than to equal  $n$ . We call this requirement a *weak Chowla condition*. The description of the corresponding sets  $S$  and  $T$  are obtained in [Theorem 14](#) and [Corollary 16](#).

We then move on to give a description of subsets  $S, T$ , with  $|S+T| \leq |S|+|T|$ , in arbitrary abelian groups provided  $S$  contains no element of order less than  $|S|+1$  (another weak Chowla condition). We show that, if the abelian group has no subgroups of order 2 or 3, then  $S$  and  $T$  are made up of arithmetic progressions with at most one missing element and periodic subsets with at most one missing element, see [Theorems 28 and 29](#). This last result is a generalization to abelian groups of [Theorem 3](#) of Rødseth and one of the authors, since it reduces to it when the group is of prime order.

The paper is organized as follows: [Section 2](#) gives some preliminary results and [Section 3](#) uses them to derive a solution to the critical pair problem for Chowla's Theorem and its strengthened version. [Section 4](#) works out some tools necessary to [Section 5](#) which is devoted to the description of 2-atoms. [Sections 6 and 7](#) make up more preliminary material for [Section 8](#) which derives the generalization to abelian groups of [Theorem 3](#).

## 2. Isoperimetric tools

In this section we recall known results on isoperimetric numbers of subsets in finite abelian groups and derive some consequences relevant to us later on. Our prime objects of concern are the 2-atoms of a subset: we shall see that they are either subgroups or Sidon sets and, in the last case, they have the largest possible isoperimetric numbers.

Let  $S$  be a subset of a finite abelian group such that  $0 \in S$ . Denote by  $\langle S \rangle$  the subgroup generated by  $S$ . For a positive integer  $k$ , we shall say that  $S$  is *k-separable* if there exists  $X \subset \langle S \rangle$  such that  $|X| \geq k$  and  $|X + S| \leq |\langle S \rangle| - k$ .

Suppose that  $S$  is *k-separable*. The *k-th isoperimetric number* of  $S$  is then defined by

$$(1) \quad \kappa_k(S) = \min\{|X + S| - |X| \mid X \subset \langle S \rangle, |X| \geq k \text{ and } |X + S| \leq |\langle S \rangle| - k\}.$$

For a *k-separable* set  $S$ , a subset  $X$  achieving the above minimum is called a *k-fragment* of  $S$ . A *k-fragment* with minimal cardinality is called a *k-atom*.

The following easy facts will be used regularly throughout the paper:

- if  $S$  is *k-separable* then  $1 \leq \kappa_{k-1}(S) \leq \kappa_k(S)$ .
- The translate  $A + g$  of a *k-atom*  $A$  is also a *k-atom*.

**Remark.** Let  $0 \in S$  be a *k-separable* subset of a finite abelian group such that  $|S| \geq k$ . Then  $\kappa_k(S) \leq k|S| - 2k + 1$ .

**Proof.** Assume the contrary. Let  $G = \langle S \rangle$ . Then we must have clearly  $|G| \geq 2k + \kappa_k(S) \geq k|S| + 2$ . Hence  $k|S| - k + 1 \leq |G| - k - 1$ . Let  $X$  be a *k-subset* of  $S$  such that  $0 \in X$ .

We have  $|S + X| \leq |S| + \sum_{x \in X \setminus \{0\}} |(S + x) \setminus S| \leq |S| + (k - 1)(|S| - 1) \leq k|S| - k + 1 < |G| - k$ .

Therefore, by (1), we have  $\kappa_k(S) \leq |S + X| - |X| \leq k|S| - 2k + 1$ , a contradiction. ■

If  $S$  is not *k-separable*, we shall put by convention  $\kappa_k(S) = k|S| - 2k + 1$  so as to have, for all  $|S| \geq k$ ,

$$(2) \quad \kappa_k(S) \leq k|S| - 2k + 1.$$

The definition of a *k-atom* implies the following lemma:

**Lemma 4.** Let  $0 \in S$  be a *k-separable* subset of a finite abelian group. Let  $A$  be a *k-atom* and suppose that  $|A| > k$ . Then, for each  $a \in A$  and  $s \in S$  we have

$$(A \setminus \{a\}) + S = A + S = A + (S \setminus \{s\}).$$

**Proof.** Let  $A' = A \setminus \{a\}$  and suppose that  $|A' + S| < |A + S|$ . Then  $|A' + S| - |A'| \leq |A + S| - 1 - |A'| = |A + S| - |A|$  contradicting the minimality of  $A$ . In other words, no element  $x$  in  $S + A$  can be uniquely written as  $x = s + a$ ,  $s \in S$  and  $a \in A$ . This means that  $A + S = A + (S \setminus \{s\})$  for each  $s \in S$ . ■

Next we recall:

**Lemma 5 ([8]).** *Let  $0 \in S$  be a  $k$ -separable subset of a finite abelian group  $G$ . Let  $F$  be a  $k$ -fragment of  $S$  and  $g \in \langle S \rangle$ . Then  $g - F$  and  $\langle S \rangle \setminus (F + S)$  are  $k$ -fragments of  $-S$ . Moreover  $\kappa_k(-S) = \kappa_k(S)$ .*

The following is a particularly useful property of  $k$ -atoms.

**Lemma 6 (The intersection property [8]).** *Let  $0 \in S$  be a  $k$ -separable subset of a finite abelian group  $G$ . Let  $A$  be a  $k$ -atom of  $S$ . Let  $F$  be a  $k$ -fragment of  $S$  such that  $A \not\subset F$ . Then  $|A \cap F| \leq k - 1$ .*

The intersection property implies easily the following description of 1-atoms.

**Corollary 7 ([7]).** *Let  $0 \in S \neq \langle S \rangle$  be a subset of a finite abelian group  $G$ . Let  $A$  be a 1-atom of  $S$  such that  $0 \in A$ . Then  $A$  is the subgroup generated by  $S \cap A$ . In particular  $\kappa_1(S)$  is a multiple of  $|A|$ .*

From these early results we can derive the following generalization of Chowla's Theorem:

**Corollary 8.** *Let  $0 \in S$  be a generating subset of a finite abelian group  $G$  such that the order of every element of  $S \setminus \{0\}$  is at least  $|S| - 1$ . Then  $\kappa_1(S) = |S| - 1$ .*

*In particular, for every nonempty subset  $X \subset G$ , we have*

$$|X + S| \geq \min\{|G|, |X| + |S| - 1\}.$$

**Proof.** If  $S$  is not 1-separable, then by definition we have  $S = \langle S \rangle$  and by the convention preceding (2) we have  $\kappa_1(S) = |S| - 1$ . Suppose therefore that  $S$  is 1-separable. Let  $A$  be a 1-atom of  $S$  containing 0. By Corollary 7,  $A$  is the subgroup of  $G$  generated by  $S \cap A$  and  $\kappa_1(S)$  is a multiple of  $|A|$ . If  $S \cap A = \{0\}$  then it follows that  $A$  is the null subgroup and we have  $\kappa_1(S) = |S + A| - |A| = |S| - 1$ . If  $S \cap A \neq \{0\}$  then by the hypothesis on the order of the elements of  $S$  we have  $\kappa_1(S) \geq |A| \geq |S| - 1$  which implies  $\kappa_1(S) = |S| - 1$  by (2).

The last inequality in the statement is a direct consequence of the definition of  $\kappa_1$ . ■

Recall that a subset  $X$  of an abelian group is a Sidon set if no two pairs of (not necessarily distinct) elements in  $X$  have the same sum. In particular  $|S \cap (S+x)| \leq 1$  for each  $x$ .

**Corollary 9.** *Let  $0 \in S$  be a  $k$ -separable subset of a finite abelian group  $G$ . Let  $A$  be a  $k$ -atom of  $S$  such that  $0 \in A$ , and suppose that  $p \geq k$  where  $p$  is the smallest prime divisor of  $|G|$ . Then either  $A$  is a subgroup of  $G$  or  $|A \cap (x+A)| \leq k-1$  for every  $x \in G$ ,  $x \neq 0$ . In particular a 2-atom of a 2-separable set is either a subgroup or a Sidon set.*

**Proof.** Without loss of generality we may suppose  $\langle S \rangle = G$ . The double inequality  $k \leq |A \cap (x+A)| < |A|$  is forbidden by Lemma 6 because  $x+A$  is also a  $k$ -atom of  $S$ . Suppose that there is  $x \in G$ ,  $x \neq 0$ , such that  $A = A+x$ . Then we have  $A = A + \langle x \rangle$ : hence  $A \cap (a+A) \supset a + \langle x \rangle$  for every  $a \in A$ . Since  $|\langle x \rangle| \geq p \geq k$ , Lemma 6 implies that we have  $A = a+A$  for every  $a \in A$  and  $A$  is a subgroup. ■

**Lemma 10.** *Let  $0 \in S$  be a generating set of a finite abelian group  $G$  of cardinality  $|S| \geq 3$ . Assume that  $|(S+g) \cap S| \leq 2$  for all  $g \in G \setminus \{0\}$ . Then  $\kappa_1(S) = |S| - 1$ . In particular  $\kappa_1(X) = |X| - 1$ , if  $X$  is a Sidon set containing 0.*

**Proof.** Suppose on the contrary that  $\kappa_1(S) \leq |S| - 2$ . Then  $S$  is 1-separable. Let  $0 \in A$  be a 1-atom of  $S$ . Then  $A$  is a nonnull subgroup of  $G$  and  $\kappa_1(S)$  is a multiple of  $|A|$ .

Let  $a \in A \setminus \{0\}$ . We have  $|S| + |A| - 2 \geq |S+A| \geq |S \cup (S+a)| \geq 2|S| - 2$  which implies  $|S| \leq |A| \leq \kappa_1(S)$ , contradicting (2). ■

The next result determines the second isoperimetric number of Sidon sets. In what follows we use the following notation. Given a subgroup  $H$  of  $G$ , by the decomposition of a subset  $S \subset G$  modulo  $H$  we mean the minimal partition of  $S$  into nonempty subsets, each one contained in a single coset of  $H$ .

**Lemma 11.** *Let  $0 \in S$  be a subset of a finite abelian group with  $|S| \geq 3$ . If  $S$  is a Sidon set then  $\kappa_2(S) = 2|S| - 3$ .*

**Proof.** Let  $G = \langle S \rangle$ . Suppose  $S$  is 2-separable, otherwise the result follows by the convention preceding (2).

Suppose against the lemma that  $\kappa_2(S) \leq 2|S| - 4$ . Let  $A$  be a 2-atom of  $S$  containing 0. We have  $|A| \geq 3$ , since otherwise  $2 + \kappa_2(S) = |S+A| \geq |S| + (|S|-1)$ , a contradiction. By Corollary 9,  $A$  is either a Sidon set or a subgroup. We have

$$\begin{aligned} |A| + 2|S| - 4 &\geq |A| + \kappa_2(S) = |S+A| = |\cup_{a \in A} (S+a)| \\ &\geq |S| + (|S|-1) + (|S|-2) = 3|S| - 3, \end{aligned}$$

which gives  $|A| \geq |S| + 1 \geq 4$ .

If  $A$  is a Sidon set, then

$$|S + A| = |\cup_{s \in S} (s + A)| \geq |A| + (|A| - 1) + (|A| - 2) \geq |A| + 2|S| - 1,$$

a contradiction.

Suppose that  $A$  is a subgroup. Then  $\kappa_2(S)$  is a multiple of  $|A|$ . In particular,  $|A| \leq 2|S| - 4$ . But then, since  $|A| \geq 4$ ,

$$|S + A| = |\cup_{a \in A} (S + a)| \geq |S| + (|S| - 1) + (|S| - 2) + (|S| - 3) \geq |A| + 2|S| - 2,$$

again a contradiction. ■

The following corollary is a result obtained in a more general context in [9]. The simple proof given here is similar to a proof given in [22].

**Corollary 12.** *Let  $S$  be a generating set of the finite abelian group  $G$  with  $0 \in S$ ,  $|S| \geq 3$  and  $\kappa_2(S) = |S| + m$ ,  $m \geq -1$ . Let  $0 \in A$  be a 2-atom of  $S$  which is not a subgroup of  $G$ . Then  $|A| \leq m + 3$ .*

**Proof.** Suppose on the contrary that  $|A| \geq m + 4$ . By Corollary 9,  $A$  is a Sidon set of  $G$ . By Lemmas 10 and 11, we have  $\kappa_1(A) = |A| - 1$  and  $\kappa_2(A) = 2|A| - 3$ .

If  $A$  generates  $G$  then  $2|A| - 3 = \kappa_2(A) \leq |S + A| - |S| = |A| + m$ , a contradiction. Therefore we may assume that  $A$  generates a proper subgroup  $Q$  of  $G$ . Let  $S = S_1 \cup \dots \cup S_j$ , where  $j \geq 2$ , be the decomposition of  $S$  modulo  $Q$ . We may assume that  $|S_1 + A| \leq \dots \leq |S_j + A|$  and, by translating  $S$ , that  $0 \in S_1$ .

If  $|S_2 + A| \leq |Q| - 1$  then,

$$2|A| - 2 = 2\kappa_1(A) \leq |A + S_1| - |S_1| + |A + S_2| - |S_2| \leq |A + S| - |S| = |A| + m,$$

against our assumption. Therefore we may assume that  $S' + A = S' + Q$  where  $S' = S \setminus S_1$ .

If  $|S_1| = 1$  then, for each 2-subset  $X$  of  $Q$ , we have  $|S + X| - |X| = |(S + X) \setminus (S_1 + X)| \leq |S' + Q| = |S' + A| = |S + A| - |S_1 + A| = |S + A| - |A|$  contradicting that  $A$  is a 2-atom of  $S$  with  $|A| \geq 3$ . Hence  $|S_1| \geq 2$ . Now if  $|S_1 + A| \leq |Q| - 2$  then  $2|A| - 3 = \kappa_2(A) \leq |S_1 + A| - |S_1| \leq |S + A| - |S| = |A| + m$ . Hence we may assume  $|S_1 + A| \geq |Q| - 1$ .

Since  $|S + Q| - 1 \leq |S' + Q| + |A + S_1| = |A + S| \leq |G| - 2$ , we must have  $S + Q \neq G$ . But in this case,

$$|S| + m = \kappa_2(S) \leq |S + Q| - |Q| \leq |S + A| + 1 - |Q| = |S'| + |A| + m - |Q| + 1.$$

It follows that  $|A| \geq |Q| - 1$ , which is impossible since  $A$  is a Sidon set. ■

Finally, the following lemma will be useful to us in ruling out the possibility that a 2-atom is a subgroup.

**Lemma 13.** *Let  $0 \in S$  be a 2-separable subset of a finite abelian group  $G$ . Suppose  $A$  is a 2-atom of  $S$  which is a subgroup of cardinality at least 3. Then there exists  $s \in S$ ,  $s \neq 0$ , such that the order of  $s$  is not more than  $\kappa_2(S)$ .*

**Proof.** Note that if  $A$  is a subgroup then  $\kappa_2(S)$  is a multiple of  $|A|$ . By Lemma 4 we have  $A + S = (A \setminus \{0\}) + S$  which implies  $(A \setminus \{0\}) \cap (-S) \neq \emptyset$ . Therefore there is a non-zero element  $s$  of  $S$  in  $A$ , and its order is not more than  $|A| \leq \kappa_2(S)$ . ■

### 3. Critical pairs under the weak Chowla condition

With the previous results we can already prove a critical pair theorem improving on the theorems of Chowla and Vosper. We first state its isoperimetric version. Recall that a subset  $S$  of an abelian group  $G$  is *periodic* if there is a nonnull subgroup  $H$  of  $G$  such that  $S + H = S$ . In other words,  $S$  is a union of cosets of  $H$ .

**Theorem 14.** *Let  $0 \in S$  be a generating 2-separable subset of a finite abelian group  $G$  such that  $\kappa_2(S) \leq |S| - 1$ . Also assume that every element of  $S \setminus \{0\}$  has order at least  $|S|$ . Then either  $S$  is an arithmetic progression or  $S \setminus \{0\}$  is periodic.*

**Proof.** By Corollary 8 we have  $\kappa_1(S) = |S| - 1$ . Let  $0 \in A$  be a 2-atom of  $S$ . Assume  $|S| \geq 3$  otherwise there is nothing to prove. By Lemma 13, the condition on the order of elements of  $S$  implies that  $A$  is not a subgroup. But then Corollary 12 implies that we have  $|A| = 2$ , say  $A = \{0, r\}$ . Assume first that  $r$  generates  $G$ . This forces  $S$  to be an arithmetic progression with difference  $r$ . Assume now that  $r$  generates a proper subgroup  $H$ . Let  $S = S_1 \cup \dots \cup S_j$ ,  $j \geq 2$  be the decomposition of  $S$  modulo  $H$ . We have  $|S| + 1 = \sum_{i=1}^j |S_i + \{0, r\}| \geq \sum_{i=1}^j \min\{|H|, |S_i| + 1\}$ , which implies  $|S_i| = |H|$  for all but one subscript. In particular  $S \cap H = \{0\}$  since otherwise  $S$  contains a nonzero element with order at most  $|H| \leq |S| - 1$ . ■

The above theorem will translate into a Chowla-type characterization of sets  $S$  and  $T$  with small sumset, this will be Corollary 16. The next result is a generalization of Theorem 2.

By the *stabilizer* of a subset  $X$  of an abelian group  $G$ , we mean the set of group elements  $x \in G$  such that  $X + x = X$ .

**Proposition 15.** *Let  $0 \in S$  be a generating subset of a finite abelian group  $G$  and let  $0 \in T$  be a subset of  $G$ . Let  $Q$  denote the stabilizer of  $S \setminus \{0\}$ . Suppose that*

$$|T + S| \leq |T| + |S| - 1 < |G| - |Q|.$$



Also assume that every element of  $S^* = S \setminus \{0\}$  has order  $\geq |S|$ . Let  $\sigma: G \rightarrow G/Q$  denote the canonical projection. One of the following holds:

- (i) either  $T \subset Q$ ,
  - (ii) or  $\sigma(S)$  and  $\sigma(T)$  are arithmetic progressions with the same difference.
- Moreover, at most one member of the decomposition of  $T$  modulo  $Q$  is not a complete coset modulo  $Q$ .

**Proof.** Either  $T = \{0\}$  and thus  $T \subset Q$  or the conditions on  $S$  imply that  $S$  is 2-separable and  $\kappa_2(S) \leq |S| - 1$ . Assume first  $Q = \{0\}$ . By [Theorem 14](#),  $S$  is an arithmetic progression. It follows easily that  $T$  is an arithmetic progression with the same difference. Assume now  $Q \neq \{0\}$ .

We have  $|\sigma(T) + \sigma(S)| \leq |\sigma(T)| + |\sigma(S)| - 1$ . Otherwise there are  $|\sigma(S)|$  cosets in  $\sigma(T) + \sigma(S)$  not present in  $\sigma(T)$ . But all these cosets are saturated in  $T + S$  (notice that  $S^*$  is  $Q$ -periodic). It follows that  $|T + S| \geq |T| + |\sigma(S)||Q| = |T| + |S| + |Q| - 1$ , a contradiction.

Moreover, the order of every element  $x \in \sigma(S) \setminus \{0\}$  is at least  $\lceil |S|/|Q| \rceil = |\sigma(S)|$ . Since the stabilizer of  $\sigma(S)^* = \sigma(S^*)$  must be  $\{0\}$ , either  $\sigma(T) = 0$  and  $T \subset Q$  or [Theorem 14](#) in  $G/Q$  implies that  $\sigma(S)$  is an arithmetic progression. It follows now that  $\sigma(T)$  is an arithmetic progression with the same difference. Since  $\sigma(T)$  contains at most a single element that is not expressible in  $G/Q$  in two different ways as a sum of one element of  $\sigma(S)$  and one element of  $\sigma(T)$ , we deduce that at most one coset modulo  $Q$  that intersects  $T$  is not included in  $T$ . ■

**Corollary 16.** Let  $0 \in S$  and  $T$  be non-empty subsets of a finite abelian group  $G$ . Suppose that

$$|S + T| \leq |S| + |T| - 1 < |H + T| - |Q|,$$

where  $Q$  denotes the stabilizer of  $S \setminus \{0\}$  and  $H$  is the subgroup of  $G$  generated by  $S$ . Also assume that every element of  $S \setminus \{0\}$  has order at least  $|S|$ . Let  $T_1 \cup T_2 \cup \dots \cup T_j$  be a decomposition of  $T$  modulo  $H$  such that

$$|T_1 + S| \leq |T_2 + S| \leq \dots \leq |T_j + S|.$$

Then  $|T_i| = |H|$  for all  $i \geq 2$ . Moreover one of the following conditions holds:

- (i)  $T_1 - T_1 \subset Q$ .
- (ii)  $\sigma(S)$  and  $\sigma(T_1)$  are arithmetic progressions with the same difference, where  $\sigma: G \rightarrow G/Q$  denotes the canonical projection.

**Proof.** By [Corollary 8](#) we have  $\kappa_1(S) = |S| - 1$ . If  $j \geq 2$  we have  $|T_2 + S| = |H|$  since otherwise,

$$2|S| - 2 = 2\kappa_1(S) \leq |S + T_1| - |T_1| + |S + T_2| - |T_2| \leq |S + T| - |T| \leq |S| - 1,$$

a contradiction. Assume first that  $0 \in T_1$ . Since  $|S + T| < |H + T| - |Q|$ , we have  $|S + T_1| < |H| - |Q|$ . We have clearly

$$|S + T_1| \leq |S| + |T_1| - 1 < |H| - |Q|.$$

By [Proposition 15](#), either  $T_1 \subset Q$  or  $\sigma(S)$  and  $\sigma(T_1)$  are arithmetic progressions with the same difference. Now, if  $0 \notin T_1$  then the same argument gives  $T_1 - T_1 \subset Q$ . ■

At the heart of the proof of [Theorem 14](#) was the claim that, under the right conditions, a 2-atom containing the zero element is of cardinality 2 or is a subgroup. In [Section 5](#) we shall find more general conditions under which we can make the same claim. Before that we need some more tools.

#### 4. The fainting technique

In this section we use a method developed in [\[22\]](#). The idea is to consider the sequence of subsets  $(S+A) \setminus S, (S+2A) \setminus (S+A), \dots, (S+iA) \setminus (S+(i-1)A), \dots$  and to claim that if  $A$  is a 2-atom of  $S$  of cardinality  $|A| > 2$ , then this sequence must decrease and faint, implying that  $S$  is a “large” subset of  $G$ .

Let  $X$  and  $Y$  be subsets of an abelian group  $G$ . For each integer  $i \geq 0$  we denote by

$$N_i(X, Y) = (X + iY) \setminus (X + (i-1)Y), \quad i > 0, \quad N_0(X, Y) = X,$$

where  $iY = \underbrace{Y + \dots + Y}_i$ .

In what follows we use the notation  $Y^* = Y \setminus \{0\}$ . We start with the two following lemmas.

**Lemma 17.** *Let  $G$  be an abelian group and let  $X, Y \subset G$  with  $0 \in X \cap Y$ . If  $N_r(X, Y) - Y^* \subset N_{r-1}(X, Y)$  for some  $r \geq 1$ , then  $N_i(X, Y) - Y^* \subset N_{i-1}(X, Y)$ , for all  $i \geq r$ .*

**Proof.** Suppose that the statement holds for all  $i, r \leq i \leq j$ , for some  $j \geq r$ , and let  $x \in N_{j+1}(X, Y)$  (if  $N_{j+1}(X, Y) = \emptyset$  there is nothing to prove). By the definition of  $N_{j+1}(X, Y)$ , there is  $z \in Y^*$  such that  $x - z \in N_j(X, Y)$ . Now, for every  $y \in Y^*$ ,  $x - y - z = (x - z) - y \in N_{j-1}(X, Y)$  which implies  $x - y \in N_j(X, Y)$ . The result follows by induction. ■

**Lemma 18.** *Let  $0 \in S$  be a 2-separable subset of a finite abelian group  $G$  and let  $0 \in A$  be a 2-atom of  $S$  with cardinality  $|A| \geq 3$  which is not a subgroup of  $G$ . Then, denoting  $A^* = A \setminus \{0\}$ ,*

$$S + A = S + (A \setminus \{a\}) \text{ for each } a \in A \text{ and } N_2(S, A) - A^* \subset N_1(S, A).$$

**Proof.** Without loss of generality  $S$  generates  $G$ . The first part of the result is just Lemma 4. Now, since  $A$  is not a subgroup, we have  $S + \langle A \rangle \neq S + A$ , otherwise we would have  $|S + \langle A \rangle| - |\langle A \rangle| < |S + A| - |A|$  in contradiction with  $A$  being a 2-atom. Therefore there exists  $x \in N_2(S, A) = (S + 2A) \setminus (S + A)$ . Recall that, by Lemma 5, the subset  $x - A$  is a 2-atom of  $-S$  and  $G \setminus (S + A)$  is a 2-fragment. Observe that  $x \in (x - A) \cap (G \setminus (S + A))$  and that  $x \in N_2(S, A)$  means  $x - A$  is not contained in  $G \setminus (S + A)$ : the intersection property of 2-atoms (Lemma 6) implies therefore that  $(x - A) \cap (G \setminus (S + A)) = \{x\}$ , but this means  $x - A^* \subset N_1(S, A)$ . ■

The following Lemma is a key tool for the proof of the main result of the next section. It says that, under some conditions, a set  $X$  verifying the statement of Lemma 18 with some other set must be a large subset of the ground group.

**Lemma 19 (The Fainting Lemma).** *Let  $G$  be a finite abelian group and let  $X, Y \subset G$  with  $0 \in X \cap Y$  and set  $m = |X + Y| - |X| - |Y|$ . Assume that  $Y$  generates  $G$  and that*

- (i)  $3 \leq |Y| \leq m + 3$  and  $\kappa_1(Y^* - y) = |Y^*| - 1 \geq 1$  for some  $y \in Y^*$ .
- (ii)  $X + Y = X + (Y \setminus \{z\})$  for each  $z \in Y$  and  $N_2(X, Y) - Y^* \subset N_1(X, Y)$ .

Then

$$|X| \geq |G| - \binom{m+4}{2}.$$

**Proof.** Since  $X + Y = X + (Y \setminus \{y\})$  for any  $y \in Y$ , we have  $X + Y = X + Y^*$  and  $X + (Y - y) = X + (Y^* - y)$ . By induction on  $i$  it is seen that  $X + i(Y - y) = X + (Y^* - y) + (i - 1)(Y - y) = X + i(Y^* - y)$  for each  $i \geq 1$ . Since  $0 \in Y$  generates  $G$ , we have  $G = X + n(Y - y) = X + n(Y^* - y)$  where  $n = |G|$ . Let  $H$  be the subgroup of  $G$  generated by  $Y^* - y$ . One can verify easily that  $H = n(Y^* - y)$ , and hence

$$(3) \quad X + H = G.$$

By Lemma 17,  $N_2(X, Y) - Y^* \subset N_1(X, Y)$  implies

$$(4) \quad N_{i+1}(X, Y) - Y^* \subset N_i(X, Y) \text{ for all } i \geq 1.$$

Fix  $y \in Y^*$  satisfying (i). Suppose that there is  $i \geq 1$  such that  $N_{i+1}(X, Y) \neq \emptyset$  and

$$(5) \quad |N_{i+1}(X, Y) - (Y^* - y)| < |N_{i+1}(X, Y)| + |Y^*| - 1.$$

Since  $\kappa_1(Y^* - y) = |Y^*| - 1$ , the inequality (5) means that  $N_{i+1}(X, Y) - (Y^* - y)$  is a union of cosets of the subgroup  $H$  generated by  $Y^* - y$ . In particular  $H \neq G$  and, by (4),  $N_i(X, Y) \supset (N_{i+1}(X, Y) - (Y^* - y)) + y$  contains a full coset of this subgroup. However, we have  $N_i(X, Y) \cap X = \emptyset$  and, by (3),  $X + H = G$ , a contradiction. Let  $\ell$  be the largest integer for which  $N_\ell(X, Y) \neq \emptyset$ . We have just shown that, for each  $i$ ,  $1 \leq i < \ell$ ,

$$|N_{i+1}(X, Y)| + |Y^*| - 1 \leq |N_{i+1}(X, Y) - (Y^* - y)| = |N_{i+1}(X, Y) - Y^*| \leq |N_i(X, Y)|.$$

Therefore,

$$(6) \quad |G| = |X| + \sum_{i=1}^{\ell} |N_i(X, Y)| \leq |X| + \sum_{i=1}^{\ell} (|N_1(X, Y)| - (i-1)(|Y| - 2)).$$

Since  $|N_1(X, Y)| = |Y| + m$  we have  $|N_2(X, Y)| \leq m + 2$ . Hence, since  $3 \leq |Y| \leq m + 3$ , the largest possible value in the right hand side of inequality (6) is taken if  $|Y| = 3$  and  $\ell = m + 3$  giving

$$|G| \leq |X| + \binom{m+4}{2},$$

as claimed. ■

We finish this set of preliminary results with the following Lemma.

**Lemma 20.** *Let  $A$  and  $S$  be subsets of a finite abelian group  $Q$ . Assume that  $|A| = 3$  and that for each  $a \in A$  we have  $S + A = S + (A \setminus \{a\})$ . Then  $3|S| \geq 2|S + A|$ .*

**Proof.** Write  $A = \{x, y, z\}$ . We have  $S + A = (x + S) \cup (y + S)$ . It follows that  $|S + A| = 2|S| - |(x + S) \cap (y + S)|$ . Furthermore we must have  $((x + S) \cup (y + S)) \setminus ((x + S) \cap (y + S)) \subset z + S$ , therefore  $|(x + S) \cap (y + S)| \geq |S|/2$ . The result now follows. ■

## 5. Description of 2-atoms

The next theorem gives the structure of the 2-atoms for not too large subsets of an abelian group.

**Theorem 21.** *Let  $G$  be a finite abelian group and let  $0 \in S$  be a generating 2-separable subset of  $G$  such that  $|S| \geq 3$  and*

$$\kappa_2(S) - |S| = m \leq 4.$$

*Let  $A$  be a 2-atom of  $S$  containing 0. If  $|S| < |G| - \binom{m+4}{2}$  then either  $|A| = 2$  or  $A$  is a subgroup of  $G$ .*

**Proof.** Suppose that the conclusion of the theorem does not hold, so that  $0 \in A$  is a 2-atom of  $S$  with  $|A| \geq 3$  which is not a subgroup. Then it follows from Corollary 9 that  $S$  is a Sidon set and then, by Lemma 11,  $\kappa_2(S) = 2|S| - 3 \geq |S|$ . In particular  $m \geq 0$ .

By Corollary 12 we have  $|A| \leq m + 3$ . Moreover,  $A^* - a$  is also a Sidon set, and Lemma 10 implies that  $A$  satisfies condition (i) of the Fainting Lemma. By Lemma 18,  $S$  and  $A$  satisfy condition (ii) of the Fainting Lemma: therefore if  $A$  generates  $G$  its conclusion must hold. In that case we have  $|S| \geq |G| - \binom{m+4}{2}$  against the hypothesis of the Theorem. Therefore  $A$  must generate a proper subgroup  $Q$  of  $G$ . Let  $S = S_1 \cup S_2 \cup \dots \cup S_t$  be the decomposition of  $S$  modulo  $Q$  and  $I = \{1, \dots, t\}$ . Put

$$\begin{aligned} U &= \{i \in I : |A + S_i| = |Q|\}, \\ V &= \{i \in I : |A + S_i| = |Q| - 1\}, \\ W &= \{i \in I : |A + S_i| \leq |Q| - 2\}, \text{ and} \\ u &= |U|, \quad v = |V|, \quad w = |W|. \end{aligned}$$

Since  $|S + A| = |S| + |A| + m$ , the decomposition of  $S + A$  modulo  $Q$  gives

$$\begin{aligned} (7) \quad |S| + |A| + m &= \sum_{i=1}^t |S_i + A| = |S| + \sum_{i=1}^t (|S_i + A| - |S_i|) \\ &\geq |S| + \sum_{i \in V \cup W} (|S_i + A| - |S_i|). \end{aligned}$$

Now, as mentioned above,  $A$  is a Sidon set and by Lemma 10, we have  $\kappa_1(A) = |A| - 1$ . Therefore  $|S_i + A| - |S_i| \geq |A| - 1$  for  $i \in V$ . Notice furthermore that Lemma 4 implies  $S_i + A = S_i + A^*$ , so that  $|S_i| \geq 2$  for each  $i \in I$ . Therefore, since by Lemma 11 we have  $\kappa_2(A) = 2|A| - 3$ , we have  $|S_i + A| - |S_i| \geq 2|A| - 3$  for  $i \in W$ . Inequality (7) gives us

$$(8) \quad 4 \geq m \geq v(|A| - 1) + w(2|A| - 3) - |A|.$$

In particular we have

$$w \leq 2 \text{ and } v \leq 3.$$

Now for any  $i \in I$  let us write

$$\delta(i) = |Q| - |S_i + A|, \quad \text{and} \quad |S_i + A| = |S_i| + |A| + m_i,$$

and, for  $J \subset I$ , put  $\delta(J) = \sum_{i \in J} \delta(i)$ . Notice that  $\delta(I) = |S + Q| - |S + A|$ ,  $\delta(U) = 0$ ,  $\delta(V) = v$  and that we have shown that  $m_i \geq |A| - 3 \geq 0$  for  $i \in W$ . We consider two cases.

**Case 1.**  $S + Q \neq G$ .

It follows that, by the minimality in the definition of a 2-atom,  $|S + Q| - |Q| \geq |S + A| - |A|$ . Therefore,

$$(9) \quad \delta(V) + \delta(W) = |S + Q| - |S + A| \geq |Q| - |A|.$$

Since  $A$  is a Sidon set we have  $|Q| - |A| \geq 4$  (for example use  $|Q| \geq |A|(|A|+1)/2$  and rule out the case  $|A|=3$  and  $|Q|=6$  by exhaustive search). By (8) and since  $\delta(V) = v \leq 3$  we have  $w \geq 1$  which in turn implies  $v \leq 2$ . Now, by the definition of  $m_i$  and  $\delta(i)$ , we have  $|Q| = |S_i| + |A| + m_i + \delta(i)$  for all  $i$  and inequality (9) can be rewritten as

$$(10) \quad \delta(W) \geq |Q| - |A| - v \geq |S_i| + m_i - v + \delta(i).$$

Let  $i \in W$ . If  $|A| \geq 4$  then  $m_i \geq |A| - 3 \geq 1$  so that  $|S_i| + m_i \geq 3$  and, if  $|A| = 3$ , then Lemma 20 and  $m_i \geq 0$  give  $3|S_i| \geq 2(|S_i + A|) \geq 2(|S_i| + |A|)$  meaning  $|S_i| \geq 6$ . In both cases inequality (10) gives  $\delta(W) > \delta(i)$ , which implies  $w \geq 2$ .

Returning to (8) it follows that  $v = 0$ ,  $w = 2$ ,  $|A| = 3$  and (7) gives  $\sum_{i \in W} m_i \leq 1$ . We may assume  $W = \{1, 2\}$ . Since  $|S_i| \geq 4$  for  $i \in W$ , inequality (10) gives

$$\delta(1) + \delta(2) \geq 4 + \delta(i), \quad i = 1, 2.$$

Hence  $\delta(i) \geq 4$  for  $i = 1, 2$ . Now since  $m_1 + m_2 \leq 1$  we have, for example,  $m_1 = 0$ . But the Fainting Lemma applied to  $S_1$  and  $A$  gives  $6 = \binom{m_1+4}{2} \geq |Q| - |S_1|$  contradicting  $|Q| - |S_1| = |A| + \delta(1) \geq 7$ .

**Case 2.**  $S + Q = G$ .

Now  $|G| = |S + A| + \delta(V) + \delta(W)$  and the hypothesis of the theorem reads

$$(11) \quad \delta(V) + \delta(W) + |A| + m \geq 1 + \binom{m+4}{2}.$$

Since  $|A| \leq m+3$  (Corollary 12), inequality (11) implies  $\delta(V) + \delta(W) \geq 4$ , giving  $w \geq 1$  since  $\delta(V) \leq 3$ .

If  $w=1$ , say  $W=\{1\}$ , then (11) translates to

$$(12) \quad |Q| - |S_1| = |A| + m_1 + \delta(1) \geq 1 + \binom{m+4}{2} + m_1 - m - \delta(V).$$

If  $m=m_1$  then we must have  $\delta(V)=0$  and the right hand side of (12) equals  $1 + \binom{m_1+4}{2}$ . If  $m_1 < m$  then  $\delta(V)=v \leq 3$  and  $m_1 \geq 0$  imply that the right hand side of (12) is again  $\geq 1 + \binom{m_1+4}{2}$ . In both cases this contradicts the Fainting Lemma applied to  $S_1$  and  $A$ .

If  $w=2$ , say  $W=\{1,2\}$ , then (8) implies  $|A|=3$ ,  $v=0$ ,  $0 \leq m_1+m_2 \leq 1$  and  $m \geq 3$ . Then, the Fainting Lemma applied to  $S_i$  and  $A$ ,  $i=1,2$ , gives  $|A|+m_i+\delta(i) \leq \binom{m_i+4}{2}$ . By adding up the two inequalities we get

$$2|A| + m_1 + m_2 + \delta(1) + \delta(2) \leq \binom{m_1+4}{2} + \binom{m_2+4}{2}.$$

But the right hand side is at most 16 while, by (11), the left hand side is at least  $\delta(W) + |A| + m - 1 \geq \binom{m+4}{2} \geq 21$  a contradiction. This completes the proof. ■

The following example shows that the result of Theorem 21 does not hold anymore if  $m=5$ .

**Example.** Take  $G = \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$  where  $q > 7$  is a prime. Consider the sets

- $S = \{0, 1, 2, 4\} \times X$  where  $|X|=4$  and  $X$  is a Sidon set in  $\mathbb{Z}/q\mathbb{Z}$  and
- $A = \{0, 1, 3\} \times \{0\}$ .

Then  $|S+A| = |S| + |A| + 5$ . The group  $G$  has only two proper subgroups  $H_1 = \mathbb{Z}/7\mathbb{Z} \times \{0\}$  and  $H_2 = \{0\} \times \mathbb{Z}/q\mathbb{Z}$  and

$$|S + H_1| = |S| + |H_1| + 5, |S + H_2| = 4q > |S| + q + 5.$$

On the other hand, if  $B = \{0, x\}$  we have

$$|S + B| \geq \begin{cases} |S| + 8 > |S| + |B| + 5, & \text{if } x \in H_1 \\ |S| + 12 > |S| + |B| + 5, & \text{if } x \notin H_1. \end{cases}$$

The last inequality being because, for any  $y \neq 0$ ,  $|X \cup (X+y)| \geq 7$  in  $\mathbb{Z}/q\mathbb{Z}$  since  $X$  is a Sidon set. Therefore subgroups and subsets of size 2 are not 2-atoms of  $S$ . Furthermore we have  $\kappa_2(S) \geq |S|+5$  since otherwise Theorem 21 would apply: therefore  $A$  is a 2-atom of  $S$  and  $\kappa_2(S) = |S|+5$ . ■

Finally, note that Theorem 21 together with Lemma 13 give a sufficient condition to rule out the possibility of a 2-atom being a subgroup.

**Corollary 22.** *Let  $G$  be a finite abelian group and let  $0 \in S$  be a generating 2-separable subset of  $G$  such that  $|S| \geq 3$  and*

$$-1 \leq \kappa_2(S) - |S| = m \leq 4.$$

*Also assume that every non zero element of  $S$  has order at least  $|S| + m + 1$ . If  $|S| < |G| - \binom{m+4}{2}$  then the 2-atoms of  $S$  have cardinality 2.*

## 6. Atoms of small sets

We next show some results about  $k$ -atoms of small sets.

**Lemma 23.** *Let  $S$  be a 4-separable generating subset of a finite abelian group such that  $0 \in S$  and  $\kappa_4(S) = |S| = 3$ . Let  $0 \in A$  be a 4-atom of  $S$ . Then  $|A| = 4$ .*

**Proof.** Let  $G = \langle S \rangle$ . Suppose that  $|A| > 4$ . We shall apply the Fainting Lemma to  $A$  and  $S$ .

We have  $\kappa_1(S^* - z) = 1$ , for all  $z \in S^*$ , since  $\kappa_1(S^* - z) > 0$ .

Take  $z \in S^*$ . By Lemma 4 we have  $A + S = A + \{0, z\} = A \cup (A + z)$ . Therefore  $((A + S) \setminus A) \subset A + z$ . Therefore  $N_1(A, S) - z \subset A = N_0(A, S)$ . It follows that

$$N_1(A, S) - S^* \subset A = N_0(A, S).$$

By Lemma 17 we have  $N_2(A, S) - S^* \subset N_1(A, S)$ .

Now we may apply the Fainting Lemma and obtain  $|A| \geq |G| - 6$ . But then  $|A + S| \geq |G| - 3$  contradicting that  $A$  is a 4-fragment of  $S$ . ■

**Lemma 24.** *Let  $0 \in S$  be a 3-separable generating subset of a finite abelian group  $G$  such that  $\kappa_3(S) = |S| = 4$ . Assume  $\gcd(|G|, 6) = 1$ . Let  $A$  be a 3-atom of  $S$  such that  $0 \in A$ . Then  $|A| = 3$ .*

**Proof.** Suppose on the contrary that  $|A| \geq 4$ . Then  $A$  is not a subgroup since otherwise  $|S + A|$  and  $\kappa_3(A)$  are multiples of  $|A|$ , so that  $|A| \leq \kappa_3(S) \leq 4$  contradicts  $\gcd(|G|, 6) = 1$ . By Corollary 9 we have

$$(13) \quad |A \cap (A + g)| \leq 2 \text{ for each } g \neq 0.$$

In particular,

$$|S| + |A| = |S + A| = |\cup_{s \in S} (s + A)| \geq |A| + (|A| - 2) + (|A| - 4),$$

which implies  $|A| \leq 5$ .



Suppose that  $A$  generates a proper subgroup  $H$  of  $G$  and let  $S = S_1 \cup \dots \cup S_j$  be the decomposition of  $S$  modulo  $H$ . By [Lemma 10](#),

$$|S_i + A| \geq \min(|H|, |S_i| + |A| - 1), \quad i = 1, \dots, j.$$

Choose  $h \in H \setminus \{0\}$ . Since  $|S_i| \leq 3$ ,

$$|H| \geq |A \cup (A + h)| \geq 2|A| - 2 \geq |S_i| + |A| - 1.$$

Therefore,  $|S + A| = \sum_{i=1}^j |S_i + A| \geq |S| + 2|A| - 2 > |S| + |A|$ , a contradiction. Hence  $\langle A \rangle = G$ .

In particular,  $|S + A| = |S| + |A| \leq |G| - 3$  implies  $\kappa_2(A) \leq \kappa_3(A) \leq |A|$ . Let  $0 \in B$  be a 2-atom of  $A$ . Since  $|A| \leq |G| - 7$ , [Theorem 21](#) implies that  $B$  is a subgroup or  $|B| = 2$ .

Suppose that  $B$  is a subgroup. Then  $|B| \geq 5$ . Let  $A = A_1 \cup \dots \cup A_j$  be the decomposition of  $A$  modulo  $B$ . We have  $(j - 1)|B| = |A + B| - |B| \leq |A| \leq 5$  which implies  $j = 2$ ,  $|A| = |B| = 5$  and  $\kappa_2(A) = |A|$ . But then  $S$  is a set with smaller cardinality than  $B$  with  $|S + A| - |S| = |A|$ , contradicting the minimality of the 2-atom.

Therefore  $|B| = 2$ . Then, using [\(13\)](#),  $|A| + |B| \geq |A + B| \geq |A| + (|A| - 2) = 2|A| - 2$  which implies  $|A| = 4$ .

Since  $S$  is 3-separable, we have  $|G \setminus (A + S)| \geq 3$ . But we must have  $|G \setminus (A + S)| > 3$ , since otherwise, by [Lemma 5](#),  $-S$  has a 3-atom  $T$  with size  $|T| = 3$ . This would imply that  $-T$  is a 3-atom of  $S$ , a contradiction. Since  $\gcd(|G|, 6) = 1$ , we must have  $|G \setminus (A + S)| = |G| - 8 \geq 5$ .

**Claim.**  $S^*$  is an arithmetic progression.

Let us write  $N_i = N_i(A, S)$ ,  $i \geq 0$ . Note that  $|N_1| = |S + A| - |A| = \kappa_3(S) = |S| = 4$ . For each subset  $X \subset S$  and for each  $i \geq 1$ , let us denote by  $N_i^X$  the set of elements  $u \in N_i$  such that  $u - X \subset N_{i-1}$  and  $X$  is a maximal subset of  $S$  with this property. By the definition,  $N_i^X = \emptyset$  whenever  $0 \in X$ . Moreover, for two different subsets  $X, Y$ , we have  $N_i^X \cap N_i^Y = \emptyset$ .

Let  $X \subset S^*$ ,  $i \geq 2$ , and  $u \in N_i^X$ , so that  $v = u - x \in N_{i-1}$  for each  $x \in X$ . Let  $Y$  be the subset of  $S^*$  such that  $v \in N_{i-1}^Y$ , implying  $v - y \in N_{i-2}$  for each  $y \in Y$ . Then  $u - y = (v - y) + x \in N_{i-1}$  implies that  $y \in X$ . We have just shown that:

$$(14) \quad \text{for } i \geq 2, \quad N_i^X - X \subset \cup_{Y \subset X} N_{i-1}^Y.$$

By [Lemma 4](#), for each  $x \in S^*$ , we have  $A + S = A + (S \setminus \{x\})$ , which implies  $N_1^{\{x\}} = \emptyset$ . By [\(14\)](#), we have  $N_i^{\{x\}} = \emptyset$  for each  $i \geq 1$  as well.

On the other hand, for each  $x \in S^*$ , inequality (13) implies

$$2 \leq |(A+x) \setminus A| \leq |N_1 \setminus N_1^{S^* \setminus \{x\}}| = 4 - |N_1^{S^* \setminus \{x\}}|,$$

so that  $|N_1^{S^* \setminus \{x\}}| \leq 2$ .

Let us now estimate  $|N_2^X|$  and  $|N_3^X|$  for  $X \subset S^*$ . Note that by Corollary 8 as  $\kappa_1(Z) = |Z| - 1$  for each subset  $0 \in Z \subset G$  with  $|Z| \leq 3$ , since the order of any nonzero element in  $G$  is at least 5. Therefore, using (14), we have

for each 2-subset  $X$  of  $S^*$ ,  $|N_2^X| + 1 \leq |N_2^X - X| \leq |N_1^X| \leq 2$  and  $N_3^X = \emptyset$ .

Since there are at most two 2-subsets of  $S^*$  for which  $|N_1^X| = 2$ , we have

$$\sum_{X \subset S^*, |X|=2} |N_2^X| \leq 2.$$

Since  $|N_1| = 4$ , then  $N_2^{S^*} - S^*$  cannot be a coset. Therefore, since  $\kappa_1(S^* - s) = |S^*| - 1$ , we have

$$(15) \quad |N_2^{S^*}| + 2 \leq |N_2^{S^*} - S^*| \leq |N_1|.$$

This implies that  $|N_2^{S^*}| \leq 2$ .

Suppose that  $|N_2^{S^*}| \leq 1$ . Then  $|N_2| = \sum_{X \subset S^*} |N_2^X| \leq 3$  and, by applying (14) with  $i = 3$  and 4, we get  $|N_3| = |N_3^{S^*}| \leq 1$  and  $|N_4| = 0$ . Therefore  $|N_2| + |N_3| \leq 4 < |G| - |S + A|$ . This means that  $Y = A \cup N_1 \cup N_2 \cup N_3 \neq G$  and  $Y + S = Y$ , which contradicts that  $S$  generates  $G$ .

Suppose now that  $|N_2^{S^*}| = 2$ . Then  $|N_2^{S^*} - S^*| = |S^*| + 1$  which implies that  $S^*$  is an arithmetic progression. This proves the claim.

Now we have  $S = \{0, a, a + d, a + 2d\}$  for some  $d \in G$ . By repeating the argument of the claim to  $S - a - d$  we get that  $\{-a - d, -d, d\}$  is an arithmetic progression as well. We cannot have  $-a - d = 0$ . Then either  $-a - d - d = 2d$  or  $-a = -2d$ . Hence either  $S = \{-4d, -3d, -2d, 0\}$  or  $S = \{0, 2d, 3d, 4d\}$ . But then  $\{0, d, 2d\}$  is a 3-atom of  $S$ . This contradiction concludes the proof. ■

## 7. Quasi-progressions

A subset  $S$  of an abelian group  $G$  will be called a *quasi-progression of difference  $r$*  if  $S$  is not a progression with difference  $r$  and if  $S$  can be obtained by deleting an element of an arithmetic progression of difference  $r$ .

**Lemma 25.** *Let  $0 \in S$  be a quasi-progression with difference  $r$  in the cyclic group  $\mathbb{Z}/n\mathbb{Z}$ . Suppose that  $S$  generates  $\mathbb{Z}/n\mathbb{Z}$  and  $|S| \geq 3$ . Let  $T \subset \mathbb{Z}/n\mathbb{Z}$  be such that  $|T| \geq 3$  and*

$$|S + T| \leq |S| + |T| \leq n - 4.$$

*Then one of the following conditions holds:*

(i)  *$T$  is either a quasi-progression with difference  $r$  or a progression with difference  $r$ .*

(ii)  *$n = 12$  and  $T$  is a coset of order 4.*

**Proof.** Put  $S = \{a, a + d, \dots, a + (j - 1)d, a + (j + 1)d, \dots, |S|d\}$ . Observe that  $S \subset \langle d \rangle + a$ . Since  $0 \in S$ , we have  $a \in \langle d \rangle$ . Then  $\mathbb{Z}/n\mathbb{Z} = \langle S \rangle = \langle d \rangle$ . Hence without loss of generality we may assume that  $j \geq \lceil |S|/2 \rceil$  and  $a = 0$ . Since  $d$  is invertible we can assume it to equal 1. Then we have

$$(16) \quad S = \{0, 1, \dots, j - 1, j + 1, \dots, |S|\}.$$

For a subset  $X \subset \mathbb{Z}/n\mathbb{Z}$  let us call *connected components* of  $X$  the maximal arithmetic progressions with difference 1 contained in  $X$ .

**Case 1.** There is a connected component  $C_1$  of  $\overline{T} = \mathbb{Z}/n\mathbb{Z} \setminus T$  such that  $|C_1| \geq |S|$ .

Then we clearly have  $|C_1 \cap (S + T)| \geq |S| - 1$ . Furthermore, since  $\{0, 1\} \subset S$ , we have  $|C \cap (S + T)| \geq 1$  for every connected component  $C$  of  $\overline{T}$ . Therefore  $\overline{T}$ , and hence  $T$ , has at most two connected components. If  $\overline{T} = C_1$  we are done. Suppose that  $\overline{T} = C_1 \cup C_2$  where  $C_2$  is the other component of  $\overline{T}$ . Since  $|T| \geq 3$ , one of the two components of  $T$ , say  $T_1$ , has cardinality  $|T_1| \geq 2$ . Since  $S$  is a quasi-progression,  $S + T_1$  is an arithmetic progression of length  $|S| + |T_1|$ . If  $|C_2| \geq 2$  then we must have either  $|C_2 \cap (T_1 + S)| \geq 2$  or  $|C_1 \cap (T_1 + S)| \geq |S|$ , a contradiction. Therefore we must have  $|C_2| = 1$ , which proves the result.

**Case 2.** For every connected component  $C$  of  $\overline{T}$ ,  $|C| < |S|$ .

It follows that every connected component  $C$  of  $\overline{S + T}$  has size 1. Then  $q = |\overline{S + T}|$  is the number of connected components of  $\overline{S + T}$ . Then  $|S + T| \geq |T| + q(j - 1) \geq |T| + q(\lceil |S|/2 \rceil - 1)$ . Since  $q \geq 4$ , we must have  $q = 4$ ,  $|S| = 4$  and  $j = 2$ , i.e.  $S = \{0, 1, 3, 4\}$ .

If  $U = \{u, u + 1, \dots, v\}$  is a connected component of  $T$ , then  $v + 1 \in (S + T) \setminus T$ . Since  $|S + T| \leq |S| + 4$ , it follows that  $T$  has exactly 4 components, and for each such component  $U$ ,  $v + 2 \notin S + T$  but  $v + 3 \in T$ . Thus  $v - 1 \notin U$ , and each component of  $T$  has exactly one element. This shows that  $|T| = 4$  and hence  $n = 12$ .

$S = \{0, 1, 3, 4\} = \{0, 1\} + \{0, 3\}$ . Now  $\overline{S + T}$  consists of 4 single-element components. It follows that  $|\overline{S + T} + \{0, 1\}| = |\overline{S + T}| + 4$ . Therefore  $-\overline{S + T} +$

$S = -\overline{S+T} + \{0, 1\} + \{0, 3\} = -\overline{S+T} + \{0, 1\}$ . It follows that  $\overline{S+T} - S$  is a union of cosets modulo the subgroup  $H$  generated by 3. Therefore  $T = G \setminus (\overline{S+T} - S)$  is an  $H$ -coset.  $\blacksquare$

**Lemma 26.** *Let  $S$  and  $T$  be subsets of  $\mathbb{Z}$  such that  $|S| = 3$ ,  $|T| = 4$  and  $|S+T| = 7$ . Then  $S$  is either a progression or a quasi-progression.*

The proof is an easy exercise.

**Lemma 27.** *Let  $S$  be a 4-separable generating subset of an abelian group  $G$  of order  $n$  such that  $0 \in S$ ,  $|S| = 3$  and  $\kappa_4(S) = |S| = 3$ . Assume moreover that  $\gcd(n, 6) = 1$ . Then  $G$  is a cyclic group and  $S$  is a quasi-progression.*

**Proof.** Put  $S = \{0, x, y\}$ . Let  $0 \in A$  be a 4-atom of  $S$ . By Lemma 23,  $|A| = 4$ . Note that  $A$  generates  $G$  since otherwise  $|A+S| \geq 2|A| > |A| + |S|$ .

We show first that every element of  $S \setminus \{0\}$  generates  $G$ . Suppose on the contrary that  $x$  generates a proper subgroup  $K$  of  $G$ . Since  $\gcd(|G|, 6) = 1$  we have  $\min\{|H|, |G/K|\} \geq 5$ .

Let  $\phi$  denote the canonical morphism from  $G$  onto  $G/K$ . Decompose  $A = A_1 \cup \dots \cup A_j$ ,  $j \geq 2$ , modulo the subgroup  $K$  and assume that  $0 \in A_1$  and  $|A_1| \leq |A_i|$ ,  $i \geq 2$ . Notice that

$$|A + \{0, x\}| = \sum_{1 \leq i \leq j} |A_i + \{0, x\}| \geq \sum_{1 \leq i \leq j} \min(|K|, |A_i| + 1) \geq |A| + j.$$

On the other hand, since  $\phi(S)$  generates  $G/K$ , we have

$$\begin{aligned} |\phi(A + S)| &= |\phi(A) + \phi(S)| \geq \min(|G/K|, |\phi(A)| + 1) > |\phi(A)| \\ &= |\phi(A + \{0, x\})|. \end{aligned}$$

Therefore,

$$|A + S| \geq |A + \{0, x\}| + \min_i |A_i| \geq |A| + j + |A_1|,$$

which implies  $j = 2$ ,  $|A_1| = 1$  and  $|A_2| = 3$ . Now  $A + S$  contains a  $K$ -decomposition  $(A_2 + \{0, x\}) \cup (A_2 + y)$  involving only two cosets. Thus  $|A + S| \geq 1 + |A_2| + |A_2 + \{0, x\}| \geq 8$ , a contradiction. Hence each of  $x$  and  $y$  generate the cyclic group  $G = \mathbb{Z}/n\mathbb{Z}$ .

Since  $|A| = 4$  and  $\gcd(|G|, 6) = 1$ , we have  $|A + \{x, y\}| \geq |A| + 1$ . Assume first that  $|A + \{x, y\}| = |A| + 1$ . Then  $A$  is an arithmetic progression with difference  $y - x$ . But  $0 \in A$  and hence  $y - x$  is invertible since  $A$  generates  $G$ . Without loss of generality we may assume  $A = \{0, 1, 2, 3\}$ . Now it comes easily that  $S$  is a quasi-progression, and the result holds.

Suppose now that  $|A + \{x, y\}| \geq |A| + 2$ . Since  $|A + \{0, x, y\}| = |A| + 3$ , we may assume that  $|A \cap (A + x)| \geq 2$ .

Now since  $x$  is invertible in  $G = \mathbb{Z}/n\mathbb{Z}$ , we may write, without loss of generality,  $S = \{0, 1, t\}$  with  $|A \cap (A + 1)| \geq 2$ . By translating and multiplying by  $-1$ , we can also assume that  $t \leq (n + 1)/2$  (notice that  $\frac{n}{2}$  is not a unit if  $n$  is even). Therefore  $A$  can be represented by two pairs of consecutive integers, and hence by a subset of 4 integers included in an interval of length  $\leq (n + 1)/2$ . On the other hand, one of the following two possibilities holds for  $S$ :

- $S$  can be represented by a subset of an integral interval of length  $\leq (n - 3)/2$ . In that case the sum  $A + S$  in  $\mathbb{Z}/n\mathbb{Z}$  has the same cardinality as the sum  $A + S$  in  $\mathbb{Z}$ , and we are done by [Lemma 26](#).
- We have  $t = (n - 1)/2$ , in which case  $S$  is included in an arithmetic progression of length 4 and difference  $2^{-1}$  (2 is invertible since  $n$  is odd) and we are done. ■

## 8. Improving both the Theorems of Chowla and of Vosper

Next we shall generalize [Theorem 14](#) to the case when  $|S + T| \leq |S| + |T|$ . Our result is also a generalization to abelian groups of [Theorem 3](#), i.e. the main result of [\[10\]](#). Let us state it first under an isoperimetric formulation. Let us call a set *quasi-periodic* if it can be obtained by deleting one element from a periodic set.

**Theorem 28.** *Let  $0 \in S$  be a generating subset of a finite abelian group  $G$  with  $\gcd(|G|, 6) = 1$  and  $4 \leq |S| \leq |G| - 7$ . Assume  $S$  to be 3-separable and  $\kappa_3(S) = |S|$ .*

*If every element of  $S \setminus \{0\}$  has order at least  $|S| + 1$ , then either  $S$  is a quasi-progression or  $S \setminus \{0\}$  is quasi-periodic.*

**Proof.** Denote by  $S^* = S \setminus \{0\}$ . Let  $0 \in A$  be a 3-atom of  $S$ .

**Claim.** *The result holds if  $A$  generates a proper subgroup  $K$  of  $G$ .*

Assume first that  $A = K$ . In this case  $\kappa_3(S)$  is a multiple of  $|A|$  and hence  $|S| \geq |A|$ . It follows that  $S \cap A = \{0\}$ , since otherwise  $A$  would contain an element of order at least  $|S| + 1$ . Now  $S + A$  is the disjoint union  $A \cup (S^* + A)$ . Hence  $|S^* + A| = |S^*| + 1$  so that  $S^*$  is quasi-periodic and the result holds.

Therefore we may assume that  $A \neq K$ . Decompose  $S = S_1 \cup \dots \cup S_j$ ,  $j \geq 2$ , modulo the subgroup  $K$ . We may assume  $0 \in S_1$  and  $|S_1 + A| \leq |S_2 + A| \leq \dots \leq |S_j + A|$ . By [Corollary 9](#) and [Lemma 10](#),

$$(17) \quad |S_i + A| \geq \min\{|K|, |S_i| + |A| - 1\}, \quad i = 1, \dots, j.$$

It follows that  $|S_i + A| = |K|$  for each  $i \geq 2$ . If  $K + S = G$  then  $|S + A| \leq |K + S| - 3$  so that  $|S_1 + A| < |K|$ . If  $K + S \neq G$  then, by the definition of  $\kappa_3$ ,  $|S + K| \geq |S| + |K| > |S| + |A| = |S + A|$  and we also have  $|S_1 + A| < |K|$ . It follows from (17) that  $|(S \setminus S_1) + A| \leq |S \setminus S_1| + 1$ , which implies that  $S \setminus S_1$  is quasi-periodic. In particular we have  $|S| \geq |K|$ , which implies that  $S_1 = \{0\}$  and  $S^*$  is quasi-periodic. This completes the proof of the claim.

We may therefore assume that  $A$  generates  $G$ . We now consider three cases.

**Case 1.**  $|A| = 3$ .

In that case  $A$  is 4-separable and  $\kappa_4(A) \leq |A|$ . If  $\kappa_4(A) < 3$ , then [Theorem 14](#) implies that  $A$  is a progression and thus  $S$  is a quasi-progression. If  $\kappa_4(A) = 3$ , then by [Lemma 27](#),  $A$  is a quasi-progression. By [Lemma 25](#),  $S$  is a quasi-progression.

**Case 2.**  $|A| = 4$ .

In that case  $A$  is 3-separable and  $\kappa_3(A) \leq |A|$ . If  $\kappa_3(A) < 4$ , then [Theorem 14](#) implies that  $A$  is a progression and thus  $S$  is a quasi-progression. If  $\kappa_3(A) = 4$  then consider a 3-atom  $B$  of  $A$  containing 0. By [Lemma 24](#),  $|B| = 3$ . Observe that  $B$  generates  $G$ , otherwise  $\gcd(|G|, 6) = 1$  implies  $|A + B| \geq |A| + 4 > |A| + |B|$ . The set  $B$  is 4-separable and  $\kappa_4(B) \leq |B|$ . If  $\kappa_4(B) < 3$  then  $B$  is a progression by [Theorem 14](#), thus  $A$ , and hence also  $S$  are quasi-progressions. If  $\kappa_4(B) = 3$ , then by [Lemma 27](#),  $B$  is a quasi-progression. By [Lemma 25](#) applied twice we conclude that  $A$  and  $S$  are quasi-progressions.

**Case 3.**  $|A| \geq 5$ .

By [Corollary 9](#), for every  $g \in G \setminus \{0\}$ , we have

$$|A \cap (A + g)| \leq 2.$$

Let  $B$  be a 2-atom of  $A$  containing 0. If  $|B| = 2$  then  $|A| + |B| \geq |A + B| \geq |A| + (|A| - 2)$  which implies  $|A| \leq 4$ . Hence we have  $|B| \geq 3$ . By [Theorem 21](#),  $B$  is a subgroup of  $G$ . We have

$$|A| + |B| \geq |A + B| \geq |A| + (|A| - 2) + (|A| - 4),$$

and hence  $|A| \geq \kappa_2(A) \geq |B| \geq 2|A| - 6$ . Since  $\gcd(|G|, 6) = 1$ , these inequalities force  $|A| = |B| = 5$ . It follows that  $|A + B| = 2|B|$  and  $A$  has a  $B$ -decomposition  $A = A_0 \cup A_1$ . We have  $|A_i| \leq 3$ , since otherwise  $|(A + x) \cap A| \geq 3$  for each  $x \in B \setminus \{0\}$ , a contradiction. Without loss of generality, we may assume  $0 \in A_0$ ,  $|A_0| = 3$  and  $|A_1| = 2$ .

Decompose  $S = S_0 \cup \dots \cup S_j$  modulo the subgroup  $B$ .

Assume first  $S + B = G$ . Let  $V = \{i : S_i + B \not\subseteq S + A\}$ . By [Corollary 8](#), we have

$$|A| = |A + S| - |S| \geq \sum_{i \in V} (|S_i + A_0| - |S_i|) \geq |V|(|A_0| - 1).$$

In particular  $|V| \leq 2$ . Note that  $|A + S| \leq |G| - 5$  since otherwise  $C = G \setminus (A + S)$  is a 3-fragment of  $-S$  and  $-C$  a 3-fragment of  $S$  with  $|C| < |A|$ . Since  $|S_i + B| - |S_i + A_0| \leq 2$ , we have  $|V| \geq 3$ , a contradiction.

Assume now  $S + B \neq G$ . Since  $|A| > 3$ , [Lemma 24](#) implies  $|S| \geq 5 = |B|$ . Therefore  $S \cap B = \{0\}$  since every element of  $S^*$  has order at least  $|S| + 1$ .

Since  $A$  generates  $G$  we have  $(A + S + B) \setminus (S + B) \neq \emptyset$ . Therefore, there is an  $i$  such that  $(A_1 + S_i) \cap (S + B) = \emptyset$ . Now  $(S + A) \setminus S$  contains the disjoint union  $W = (A_0 \setminus \{0\}) \cup (S_i + A_1)$ . But  $|W| \geq |A| - 1$ . It follows easily that  $S \setminus \{0\}$  is periodic or quasi-periodic. The first possibility is excluded by the condition  $\kappa_3(S) = |S|$ . ■

[Theorem 28](#) translates into a characterization of subsets  $S$  and  $T$  such that  $|S + T| \leq |S| + |T|$  under some Chowla-type conditions. This was our final goal in this paper.

**Theorem 29.** *Let  $G$  be a finite abelian group with  $\gcd(|G|, 6) = 1$ .*

*Let  $0 \in S$  be a generating subset of  $G$  such that  $|S| \geq 4$  and every element in  $S^*$  has order at least  $|S| + 1$ . Let  $Q$  be a maximal subgroup such that  $|S^* + Q| - |S^*| \leq 1$  and let  $\sigma : G \rightarrow G/Q$  denotes the canonical projection.*

*Let  $T$  be a subset of  $G$  such that  $|T| \geq 3$  and suppose that  $|S + T| = |S| + |T| \leq |G| - 4$ . Then the following holds:*

- *If  $Q = \{0\}$  then  $S$  and  $T$  are progressions or quasi-progressions with the same difference.*
- *If  $Q \neq \{0\}$ ,  $|\sigma(T)| \geq 2$ , and  $|\sigma(S + T)| < |G|/|Q| - 1$  then  $\sigma(S)$  and  $\sigma(T)$  are arithmetic progressions with the same difference. Moreover we have  $|(T \setminus T_1) + Q| \leq |T \setminus T_1| + 1$  where  $T_1$  is a subset of  $T$  such that  $\sigma(T_1)$  is a single, extremal element of the progression  $\sigma(T)$ . Furthermore if  $0$  is not an extremal element of the progression  $\sigma(S)$ , then  $|T + Q| \leq |T| + 1$ .*

**Proof.**

**Case 1:**  $Q = \{0\}$ .

The conditions  $|S| + |T| = |S + T| \leq |G| - 4$  and  $|T| \geq 3$  imply that  $S$  is 3-separable and that  $\kappa_3(S) \leq |S|$ . By [Theorems 14 and 28](#),  $S$  is an arithmetic progression or quasi-progression. By [Lemma 25](#) it follows that  $T$  is an arithmetic progression or quasi-progression with the same difference.

**Case 2:**  $Q \neq \{0\}$ .

Since  $|S| \geq q = |Q|$  and each element in  $S$  has order at least  $|S| + 1$ , we have  $S \cap Q = \{0\}$ . In particular,  $\sigma(S)^* = \sigma(S^*)$ . Note that each element in  $\sigma(S)^*$  has order at least  $(|S| + 1)/q \geq |\sigma(S)| - 1 + 1/q$ .

Corollary 8 implies

$$(18) \quad |\sigma(S) + \sigma(T)| \geq |\sigma(S)| + |\sigma(T)| - 1.$$

First notice that

$$(19) \quad \Sigma = (S^* + T) \setminus (T + Q) \text{ is } Q\text{-periodic.}$$

This holds clearly if  $S^*$  is  $Q$ -periodic. So we may assume  $|S^* + Q| - |S^*| = 1$ . Let us then denote by  $S_1$  the unique subset of  $S$  of size  $|Q| - 1$  in the decomposition of  $S$  modulo  $Q$ . If  $\Sigma$  is not  $Q$ -periodic then some  $Q$ -coset must have a trace  $U$  of size  $|Q| - 1$  on the set  $\Sigma$ , and we have  $U = S_1 + T'$  where  $T' = (a + Q) \cap T$ , for some  $a$ . Since  $|S_1| = |Q| - 1$  we must have  $|T'| = 1$ . Note also that  $\sigma(S_1) + \sigma(T')$  cannot be obtained in any other way as a sum of an element of  $\sigma(S)$  and of an element of  $\sigma(T)$ , therefore  $(S_1 + T') \cap (S + (T \setminus T')) = \emptyset$ , hence  $|S + (T \setminus T')| < |S| + |T \setminus T'| - 1$ , but this contradicts  $\kappa_1(S) = |S| - 1$  (Corollary 8) and proves (19).

We therefore have:

$$(20) \quad |\sigma(S) + \sigma(T)| = |\sigma(S)| + |\sigma(T)| - 1$$

otherwise (18) implies  $|S + T| \geq |T| + |\sigma(S)||Q| > |T| + |S|$  a contradiction.

By our assumptions,  $Q$  is a maximal subgroup such that  $|S^* + Q| - |S^*| \leq 1$ . This is easily seen to imply that  $\sigma(S^*)$  is not periodic. Moreover, each element in  $\sigma(S)^*$  has order at least  $(|S| + 1)/q \geq |\sigma(S)| - 1 + 1/q$ . Then, by Proposition 15,  $\sigma(S)$  and  $\sigma(T)$  are arithmetic progressions with the same common difference  $d$ . Since  $-d$  is also a difference of  $\sigma(S)$  and  $\sigma(T)$ , we may assume without loss of generality that the terminal element  $u$  of  $\sigma(S)$  is not 0. Therefore if we set  $S' = \sigma^{-1}(u) \cap S$  we have  $|S'| \geq |Q| - 1$ . Let us suppose, without loss of generality, that the initial element of  $\sigma(T)$  is 0.

To conclude we prove the following:

**Claim.** *Let  $B \subset G$  be such that  $\sigma(B)$  is an arithmetic progression of difference  $d$  and initial element 0, and with  $|\sigma(B)| \leq |\sigma(T)|$ . Let  $B_1 = \sigma^{-1}(0) \cap B$  and set  $B_2 = B \setminus B_1$ . Suppose that  $|S + B| = |S| + |B| - \varepsilon$  where  $\varepsilon$  equals 0 or 1. Then  $|B_2 + Q| \leq |B_2| + 1 - \varepsilon$ .*

The claim is proved by induction on  $t = |\sigma(B)|$ . If  $|\sigma(B)| = 1$  then  $B_2 = \emptyset$  and there is nothing to prove. If  $|\sigma(B)| \geq 2$ , then let  $b$  be the terminal element



of  $\sigma(B)$  and set  $B' = \sigma^{-1}(b) \cap B$ . Then, since  $\kappa_1(S) = |S| - 1$ :

$$(21) \quad \begin{aligned} |S| + |B| - |B'| - 1 &\leq |S + (B \setminus B')| \leq |S + B| - |S' + B'| \\ &= |S| + |B| - \varepsilon - |S' + B'|. \end{aligned}$$

From (21) we obtain, since  $|S'| \geq |Q| - 1$ ,  $|B'| \geq |S' + B'| - 1 \geq |Q| - 2 \geq 5 - 2 = 3$ . Hence, since  $|S'| + |B'| > |Q|$ , we have  $S' + B' = Q$ . Applying (21) again we also have  $|S' + B'| \leq |B'| + 1 - \varepsilon$ , hence

$$(22) \quad |B'| \geq |Q| - 1 + \varepsilon.$$

By (21) we have

$$|S + (B \setminus B')| \leq |S| + |B \setminus B'| + (|B'| - |S' + B'|) - \varepsilon.$$

Now either  $|B'| - |S' + B'| = 0$  and the result holds by the induction hypothesis applied to  $B \setminus B'$ , or  $|B'| - |S' + B'| = -1$ . But in this case (22) implies  $\varepsilon = 0$ , and the result again holds by applying the induction hypothesis to  $B \setminus B'$  with  $|S + (B \setminus B')| \leq |S| + |B \setminus B'| - 1$ . This proves the claim and the theorem with  $T_1 = \sigma^{-1}(0) \cap T$ .

To complete the proof, consider the case when 0 is not an extremity of  $\sigma(S)$ . Then the claim applied to  $S$  and  $T$  by permuting initial and terminal elements of  $\sigma(S)$  and  $\sigma(T)$  gives  $|T_1| \geq |Q| - 1$ , so that every non-empty intersection of  $T$  with a coset of  $Q$  has cardinality at least  $|Q| - 1$ . In particular,  $S^* + T$  is  $Q$ -periodic. Since 0 is not an extremal element of  $\sigma(S)$  and  $|\sigma(T)| \geq 2$  we obtain  $\sigma(S^*) + \sigma(T) = \sigma(S) + \sigma(T)$  so that  $S + T$  is also  $Q$ -periodic. By (20) we have  $|T + Q| \leq |T| + 1$ . ■

**Remark.** One may wonder what happens if we remove from Theorem 29 the hypothesis  $|\sigma(S + T)| < |G|/|Q| - 1$ . Then the sets  $\sigma(S)$  and  $\sigma(T)$  are not necessarily arithmetic progressions any more. However, one may show that there again exists  $T_1 \subset T$ , such that  $|\sigma(T_1)| \leq 1$  and  $T \setminus T_1$  is  $Q$ -periodic or  $Q$ -quasi-periodic. We leave out the details.

### Acknowledgements

The authors are grateful to an anonymous referee for his/her careful reading and very valuable suggestions and remarks.

## References

- [1] N. ALON, M. B. NATHANSON and I. Z. RUZSA: The polynomial method and restricted sums of congruence classes, *J. Number Theory* **56** (1996), 404–417.
- [2] A. L. CAUCHY: Recherches sur les nombres, *J. Ecole Polytechnique* **9** (1813), 99–116.
- [3] S. CHOWLA: A theorem on the addition of residue classes: applications to the number  $\Gamma(k)$  in Waring’s problem, *Proc. Indian Acad. Sci.* **2** (1935), 242–243.
- [4] H. DAVENPORT: On the addition of residue classes, *J. London Math. Soc.* **10** (1935), 30–32.
- [5] J.-M. DESHOUILLERS and G. A. FREIMAN: A step beyond Kneser’s Theorem for abelian finite groups, *Proc. London Math. Soc.* (3) **86**(1) (2003), 1–28.
- [6] B. GREEN and I. Z. RUZSA: Sets with small sumset and rectification, *Bull. London Math. Soc.* **38** (2006), 43–52.
- [7] Y. O. HAMIDOUNE: On the connectivity of Cayley digraphs, *Europ. J. Combinatorics* **5** (1984), 309–312.
- [8] Y. O. HAMIDOUNE: An isoperimetric method in Additive Theory, *J. Algebra* **179** (1996), 622–630.
- [9] Y. O. HAMIDOUNE: Some results in additive number theory I: The critical pair theory, *Acta Arithmetica* **96** (2000), 97–119.
- [10] Y. O. HAMIDOUNE and Ø. J. RØDSETH: An inverse theorem modulo  $p$ , *Acta Arithmetica* **92** (2000), 251–262.
- [11] Y. O. HAMIDOUNE, A. S. LLADÓ and O. SERRA: On subsets with a small product in torsion-free groups, *Combinatorica* **18**(4) (1998), 529–540.
- [12] Y. O. HAMIDOUNE: On small subset product in a group. Structure Theory of set-addition; *Astérisque* no. **258**(xiv-xv) (1999), 281–308.
- [13] Y. O. HAMIDOUNE and A. PLAGNE: A new critical pair theorem applied to sum-free sets in abelian groups, *Commentarii Mathematici Helvetici* **79**(1) (2004), 183–207.
- [14] Y. O. HAMIDOUNE, O. SERRA and G. ZÉMOR: On the critical pair theory in  $\mathbb{Z}/p\mathbb{Z}$ , *Acta Arithmetica* **121** (2006), 99–115.
- [15] G. A. FREIMAN: *Foundations of a structural theory of set addition*, Transl. Math. Monographs **37**, Amer. Math. Soc., Providence, RI, 1973.
- [16] GY. KÁROLYI: An inverse theorem for the restricted set addition in abelian groups, *J. Algebra* **290** (2005), 557–593.
- [17] GY. KÁROLYI: Cauchy–Davenport theorem in group extensions, *Enseign. Math.* (2) **51**(3-4) (2005), 239–254.
- [18] J. H. B. KEMPERMAN: On small sumsets in an abelian group, *Acta Math.* **103** (1960), 63–88.
- [19] M. KNESER: Summenmengen in lokalkompakten abelschen Gruppen, *Math. Zeit.* **66** (1956), 88–110.
- [20] H. B. MANN: An addition theorem of abelian groups for sets of elements, *Proc. Amer. Math. Soc.* **4** (1953), 423.
- [21] M. B. NATHANSON: *Additive Number Theory; Inverse problems and the geometry of sumsets*, Grad. Texts in Math. **165**, Springer, 1996.
- [22] O. SERRA and G. ZÉMOR: On a generalization of a theorem by Vosper, *Integers Electr. J. Comb. Num. Th.* **0** (200), A10 (electronic). <http://www.integers-ejcnt.org/vol0.html>
- [23] T. TAO and V. H. VU: *Additive Combinatorics*, Cambridge Studies in Advanced Mathematics **105** (2006), Cambridge University Press.

- [24] G. VOSPER: The critical pairs of subsets of a group of prime order, *J. London Math. Soc.* **31** (1956), 200–205.
- [25] G. ZÉMOR: A generalization to noncommutative groups of a theorem of Mann, *Discrete Math.* **126(1-3)** (1994), 365–372.

Yahya O. Hamidoune

*E. Combinatoire et Optimisation*  
*Université Pierre et Marie Curie (Paris 6)*  
*4, place Jussieu*  
*75252 Paris Cedex 05*  
*France*  
[hamidoune@math.jussieu.fr](mailto:hamidoune@math.jussieu.fr)

Oriol Serra

*Dept. Matemàtica Aplicada 4*  
*Universitat Politècnica de Catalunya*  
*Jordi Girona, 1*  
*E-08034 Barcelona*  
*Spain*  
[oriol.serra@upc.edu](mailto:oriol.serra@upc.edu)

Gilles Zémor

*Institut de Mathématiques de Bordeaux*  
*Université Bordeaux 1*  
*351, cours de la Liberation*  
*F-33405 Talence Cedex*  
*France*  
[zemor@math.u-bordeaux1.fr](mailto:zemor@math.u-bordeaux1.fr)